Sect 3.1 to 3.5

Sec 3.1, 3.2: General Solution of Second Order Linear Homogeneous Equations

Motivation. Find the solution to the i.v.p

$$y'' - 5y' = 0, \quad y(t_0) = 3, \quad y'(t_0) = 7.$$

Sol. Setting $u = y'$ we have $u' - 5u = 0$ which is equivalent to $u' = 5u$. By separable equations, one sees that $u(t) = Ce^{5t}$. Integrating we have

$u = y' = Ce^{5t}$

$y = \int Ce^{5t}dt = \frac{C}{5}e^{5t} + C_1 = C_2 e^{5t} + C_1$ ← 2 sol$^{\underline{n}}$

$$y(t) = c_1 \cdot e^{5t} + c_2 \cdot 1$$

So the general solution is a linear combination of two functions $y_1(t) = e^{5t}$ and $y_2(t) = 1$. Thus, the general solution can be written as

$$y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$$

where y_1 and y_2 are both solutions to the equation. To solve the i.v.p. we need solve the system of equations

$$\begin{cases} c_1 \cdot y_1(t_0) + c_2 \cdot y_2(t_0) = 3 \\ c_1 \cdot y_1'(t_0) + c_2 \cdot y_2'(t_0) = 7 \end{cases}$$

i.e., we need to guarantee the existence of a unique solution to the matrix system

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

So we need to impose the restriction

$$\det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \neq 0$$

This motives the following definition.

A **second order linear homogeneous equation** in standard form is given by

$$y'' + p(t)y' + q(t)y = 0, \quad a < t < b.$$

The set $\{y_1(t), y_2(t)\}$ is a **fundamental set** for this 2^{nd} order differential equation if:
i) Each function individually satisfy the differential equation.
ii) The Wronskian of $y_1(t)$ and $y_2(t)$

i) Each function individually satisfy the differential equation.
ii) The Wronskian of $y_1(t)$ and $y_2(t)$

$$W(t) := \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$

is nonzero for every t in the open interval (a, b).

Theorem: If $\{y_1(t), y_2(t)\}$ is a fundamental set for the 2nd order linear homogeneous equation

$$y'' + p(t)y' + q(t)y = 0, \quad a < t < b$$

Then the general solution to this equation is given by

$$y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$$

where c_1 and c_2 are arbitrary constants.

Ex 1: Consider the initial value problem Linear homogeneous

$$y'' - \frac{1}{t}y' - \frac{3}{t^2}y = 0, \quad y(1) = 4, \quad y'(1) = 8, \quad 0 < t < \infty$$

(a) Show that $y_1(t) = t^3$ and $y_2(t) = t^{-1}$ form a fundamental set.
(b) Solve the initial value problem.

(a) verify

y	y'	y''	Sub $y'' - \frac{1}{t}y' - \frac{3}{t^2}y = 0$
$y_1 = t^3$	$y_1' = 3t^2$	$y_1'' = 6t$	$(6t) - \frac{1}{t}(3t^2) - \frac{3}{t^2}(t^3) = 0$ check $\Rightarrow 6t - 3t - 3t = 0$ ✓
$y_2 = t^{-1}$	$y_2' = -t^{-2}$	$y_2'' = 2t^{-3}$	$(2t^{-3}) - \frac{1}{t}(-t^{-2}) - \frac{3}{t^2}(t^{-1})$ $= \underbrace{2t^{-3} + t^{-3} - 3t^{-3}}_{} = 0$

$\{t^3, t^{-1}\}$ sol'ns

(a) $W = \begin{vmatrix} t^3 & t^{-1} \\ 3t^2 & -t^{-2} \end{vmatrix} = -t - 3t = -4t \neq 0 \quad (t>0)$

$\{y_1 = t^3, y_2 = t^{-1}\}$: a fundamental set of sol^ns.

G.S. : $\boxed{y = C_1 t^3 + C_2 t^{-1}}$

(b) IVP. $\quad y(1) = 4$

$\Rightarrow C_1, C_2 = ? \quad y'(1) = 8$

$y(1) = C_1 + C_2 = 4$

$y'(1) = 3C_1 - C_2 = 8$

DNF

Existence and Uniqueness Theorem for second order linear IVP

Let $p(t)$, $q(t)$, and $g(t)$ be continuous functions on the interval (a, b), and let t_0 be in (a, b). Then the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

has a unique solution defined on the entire interval (a, b).

(Theorem 3.1 page 111 textbook)

Ex: Determined the largest t-interval in which we can guarantee the existence and uniqueness of a solution of the IVP

linear

- $ty'' + cos(t)y + t^2y = t, \quad y(-1) = 1, y'(-1) = 2$

① Standard Form

$(\div t) \quad y'' + \underbrace{\left(\frac{\cos(t)}{t} + t\right)}_{q(t)} y = \underbrace{1}_{g(t)} \qquad p(t) = 0$

② Check the continuity on p, q, and g $\qquad \left\{ \begin{array}{l} p = 0 \\ q = \frac{\cos(t)}{t} + t \\ g = 1 \end{array} \right\}$ are continuous for all $t \neq 0$

③ Find one t-interval including $t_0 = -1$

IVP has a unique sol^n in $\underbrace{t < 0}$ ← Domain of sol^n

- $ty'' + cos(t)y + t^2y = t, \quad y(1) = 1,\, y'(1) = 2$

↙ DE is non-linear

Ex: Can you apply the same result to the IVP $ty'' + cos(y)y + t^2y = t, \quad y(-1) = 1,\, y'(-1) = 2$? Explain your answer.

Abel's Formula for 2nd Order Linear System

Let $y_1(t)$ and $y_2(t)$ be solutions of the 2nd order linear homogeneous equation

$$\boxed{y'' + p(t)y' + q(t)y = 0,} \quad a < t < b,$$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \neq 0$

where p and q are continuous functions in (a, b). Let $W(t)$ be the Wronskian of $y_1(t)$ and $y_2(t)$ and let t_0 be a point in the open interval (a, b). Then

$$W(t) = W(t_0) \cdot \exp\Big(-\int_{t_0}^{t} p(s)\, ds\Big)$$

$W' + p(t)W = 0$

$W = Ce^{\int -p(t)dt}$

solve W_0

$W(t_0) \neq 0 \iff W(t) \neq 0$ for all $a < t < b$

use sample number t_0 to get $W(t_0) \neq 0$? DNF

Sec 3.3, 3.4, 3.5: Constant Coefficient Homogeneous Equations

It has the form

$$ay'' + by' + cy = \boxed{0.}$$

$t \quad t^2 \quad t^3$

$Y' = \underbrace{A}\, Y \leftarrow$ constant matrix

$\Rightarrow$ use E-pairs

where $\boxed{a, b, c \text{ are constants,}}$ $a \neq 0$.

Abel's idea: What if $f(t) = \boxed{e^{\lambda t}}$ is a solution?

candidate to the sol$\underline{n}$

$\Rightarrow \lambda$

$y = e^{\lambda t}$
$y' = \lambda e^{\lambda t}$
$y'' = \lambda^2 \cdot e^{\lambda t}$

•) plug $y = e^{\lambda t}$ into $ay'' + by' + cy = 0$

$a(\lambda^2 e^{\lambda t}) + b(\lambda e^{\lambda t}) + c(e^{\lambda t}) = 0$

$e^{\lambda t}(a\lambda^2 + b\lambda + c) = 0$

$\div e^{\lambda t} \neq 0 \qquad a\lambda^2 + b\lambda + c = 0 \leftarrow$ characteristic equation

λ: a root of characteristic equation

$\Rightarrow$ sol$\underline{n}$ $y = e^{\lambda t}$

Associated to $ay'' + by' + cy = 0$ there is a characteristic polynomial equation:

$$a\lambda^2 + b\lambda + c = 0.$$

For example, solve $y'' - 2y' - 8y = 0$

① Charastic Eq.

$$1 \cdot \lambda^2 - 2\lambda^1 - 8\lambda^0 = 0$$

$$\lambda^2 - 2\lambda - 8 = 0$$

② charestk roots = $(\lambda - 4)(\lambda + 2) = 0$

$$\lambda = 4 \quad \lambda = -2$$

③ 2 sol$\underline{n}$: $y_1 = e^{4t} \quad y_2 = e^{-2t}$

④ $W = \begin{vmatrix} e^{4t} & e^{-2t} \\ 4e^{4t} & -2e^{-2t} \end{vmatrix} \underset{t_0=0}{\overset{\uparrow}{=}} \begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix}$

DNF

Associated to $ay'' + by' + cy = 0$ there is a characteristic polynomial equation:

$$a\lambda^2 + b\lambda + c = 0.$$

Recall that the roots of this polynomial equation are

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Theorem Let $\Delta = b^2 - 4ac$ be the discriminant of the characteristic polynomial equation.

Case 1. If $\Delta > 0$, the characteristic polynomial equation has two distinct real roots, say $\lambda_1 \neq \lambda_2$. Then $\{\mathbf{e}^{\lambda_1 t}, \mathbf{e}^{\lambda_2 t}\}$ is a fundamental set.

Case 2. If $\Delta = 0$, the characteristic polynomial equation has one repeated real root, say $\lambda_1 = \lambda_2$. Then $\{\mathbf{e}^{\lambda_1 t}, t\mathbf{e}^{\lambda_1 t}\}$ is a fundamental set.

Case 3. If $\Delta < 0$, the characteristic polynomial equation has two complex conjugated roots, say $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$. Then the functions

$$y_1(t) = \frac{\mathbf{e}^{\lambda_1 t} + \mathbf{e}^{\lambda_2 t}}{2} = \mathbf{e}^{\alpha t}\cos(\beta t) \qquad \text{and} \qquad y_2(t) = \frac{\mathbf{e}^{\lambda_1 t} - \mathbf{e}^{\lambda_2 t}}{2i} = \mathbf{e}^{\alpha t}\sin(\beta t)$$

form a fundamental set.

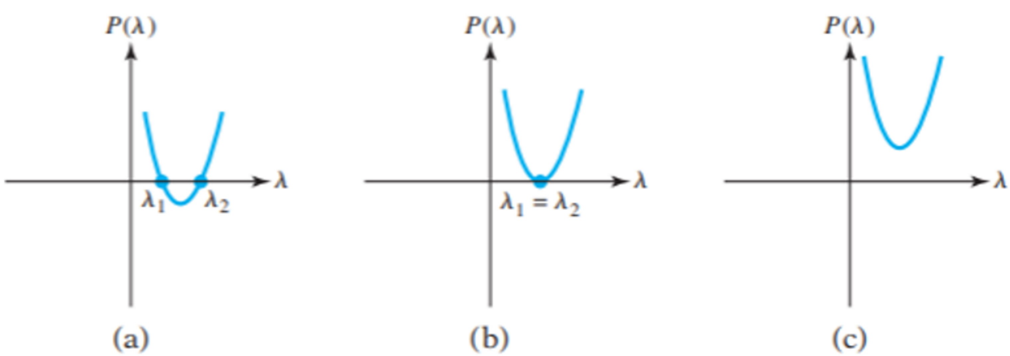

FIGURE 3.4

Three possibilities for the graph of $P(\lambda) = a\lambda^2 + b\lambda + c, a > 0$.
(a) The characteristic equation has two real distinct roots, λ_1 and λ_2.
(b) The characteristic equation has a single repeated real root, λ_1.
(c) The characteristic equation has two complex roots but no real roots.

(From page 123, textbook)

Use **Euler's Formula** for $e^{i\lambda t}$

$$e^{i\lambda t} = \cos(\lambda t) + i \sin(\lambda t)$$

and the **superposition principle** (theo 3.2, page 116 textbook)

> Let $y_1(t)$ and $y_2(t)$ be any two solutions of
> $$y'' + p(t)y' + q(t)y = 0$$
> defined on the interval (a, b). Then, for any constants c_1 and c_2, the linear combination
> $$y(t) = c_1 y_1(t) + c_2 y_2(t)$$
> is also a solution on (a, b).

to solve the following i.v.p. and describe the behavior of the solution $y(t)$ as $t \to \infty$.

make sure there are constant coeff.

1. $y'' + 5y' + 6y = 0, \quad y(0) = 1, \quad y'(0) = -1.$

① Char. eqn & roots:

$$\lambda^2 + 5\lambda + 6 = (\lambda+2)(\lambda+3) = 0$$

$$\lambda = -2, -3 \qquad \text{(distinct real)}$$

② 2 sol$^{\underline{n}}$s

$$e^{-2t} \qquad e^{-3t}$$

$$y = C_1 e^{-2t} + C_2 e^{-3t}$$

③ IVP C_1, C_2

$$y' = -2C_1 e^{-2t} - 3C_2 e^{-3t}$$

$$y(0) = C_1 + C_2 = 1 \qquad 2C_1 + 2C_2 = 2$$

$$y'(0) = \underbrace{-2C_1 - 3C_2 = -1} \qquad \underbrace{-2C_1 - 3C_2 = -1}$$

$$-C_2 = 1 \qquad C_2 = -1$$

$$C_1 = 2$$

$$\boxed{y = 2e^{-2t} - e^{-3t}}$$

4 $\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \left(\overbrace{\cancel{2e^{-2t}}}^{\frac{1}{e^{2t}}} \xrightarrow{} 0 \;\; \cancel{-e^{-3t}} \to 0 \right) = 0$

2. $25y'' + 20y' + 4y = 0, \quad y(5) = 4e^{-2}, \quad y'(5) = -(3/5)e^{-2}.$

3. $y'' + 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = 2.$

Constant Coefficient 2^{nd} order Linear Homogeneous Equations(Continuation)

Ex The graph of the solution of the differential equation $4y'' + 4y' + y = 0$ passes through the points $(1, e^{-1/2})$ and $(2, 0)$. Determine $y(0)$ and $y'(0)$.

① sol^n $4\lambda^2+4\lambda+1 = (2\lambda+1)^2=0$ $\lambda=-1/2\,;-1/2$

$y_1 = C_1e^{-1/2t} + C_2te^{-1/2t}$

② $y(1)=C_1e^{-1/2}+C_2e^{-1/2} = e^{-1/2} = C_1+C_2$

$y(2) = C_1\cancel{e^{-1}} + C_2 \cancel{2e^{-1}} = 0 \Rightarrow \underline{C_1+2C_2=0}$ $C_2=-1$

$-C_2 = 1$ $C_1 = 2$

$y = 2e^{-1/2t} - te^{-1/2t}$

$y' = -e^{-1/2t} - \left(e^{-1/2t} + (-\tfrac{1}{2})te^{-1/2t}\right)$

$\boxed{y(0)=2}$ $\boxed{y'(0) = -1-1=-2}$

Ex Consider the i.v.p.

$$y'' + by' + cy = 0, \quad y(0) = 1, \quad y'(0) = \mathbf{r},$$

where b, c, and $\mathbf{r}$ are constants. It is known that one solution of the differential equation is $y_1(t) = e^{-3t}$ and that the solution to the i.v.p. satisfies $\lim y(t) = 2$. Determine the constants b

$$y'' + by' + cy = 0, \quad y(0) = 1, \quad y'(0) = r,$$

where b, c, and $\mathbf{r}$ are constants. It is known that one solution of the differential equation is $y_1(t) = \mathbf{e}^{-3t}$ and that the solution to the i.v.p. satisfies $\lim_{t\to\infty} y(t) = 2$. Determine the constants b, c and $\mathbf{r}$.

① From $y_1(t) = e^{-3t}$:

$\lambda_1 = -3$: roots of char. eqn : $\lambda^2 + b\lambda + c = 0$

• $\lambda_2 = -3$ ← repeated roots

$y = C_1 e^{-3t} + C_2 t e^{-3t}$

$\lim_{t\to\infty} \cancel{C_1 e^{-3t}} + \cancel{C_2 \frac{t}{e^{3t}}} = 2$ $(\lambda_2 \neq -3)$

• $\lambda_2 = (+)$: $y_1 = C_1 e^{-3t} + C_2 e^{\lambda_2 t}$

$\lim_{t\to\infty} \cancel{C_1 e^{-3t}} + \cancel{C_2 e^{\lambda_2 t}} = 2$ $(\lambda_2 \neq (+))$

• $\lambda_2 = (-)$ $(\lambda \neq -3)$ (Two distinct roots)

$\lim_{t\to\infty} (y = \cancel{C_1 e^{-3t}} + \cancel{C_2 e^{\lambda_2 t}}) \neq 2$ $(\lambda_2 \neq (-))$

$\boxed{\lambda_2 = 0}$

1) Solve $16y'' + 40y' + 25y = 0$

characteristic Eqn & Two roots

$$16r^2 + 40r + 25 = (4r+5)^2 = 0$$

$$r = -5/4, \ -5/4$$

$$y_1 = e^{-5/4 t} \qquad y_2 = t\, e^{-5/4 t}$$

General Soln $y = C_1 e^{-5/4 t} + C_2 t e^{-5/2 t}$

2) $y' - 2y = 0$ ← Want Coeff

$r' - 2 = 0 \qquad r = 2$

Soln $\boxed{y = c \cdot e^{2t}}$

3.5 Complex Roots (Case III)

Given $ay'' + by' + cy = 0$, two complex roots of characteristic polynomial are

$$r_{1,2} = -\frac{b}{2a} \pm i\,\frac{\sqrt{4ac - b^2}}{2a}$$

We rewrite

$$r_1, r_2 = \alpha \pm i\beta$$

Two solutions are

$$y_1(t) = e^{(\alpha + i\beta)t} \qquad y_2(t) = e^{(\alpha - i\beta)t}$$

$$y_1(t) = \frac{e^{r_1 t} + e^{r_2 t}}{2} = e^{\alpha t}\cos(\beta t) \quad \text{and} \quad y_2(t) = \frac{e^{r_1 t} - e^{r_2 t}}{2i} = e^{\alpha t}$$ DNF